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Weak inertial-wave turbulence theory

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A weak wave turbulence theory is established for incompressible fluids under rapid rotation using a helicity decomposition, and the kinetic equations for energy E and helicity H are derived for three-wave coupling. As expected, nonlinear interactions of inertial waves lead to two-dimensional behavior of the turbulence with a transfer of energy and helicity mainly in the direction perpendicular to the rotation axis. For such a turbulence, we find, analytically, the anisotropic spectra $E\sim k_\perp^{-5/2}k_\parallel^{-1/2}$, $H\sim k_\perp^{-3/2}k_\parallel^{-1/2}$, and we prove that the energy cascade is to small scales. At lowest order, the wave theory does not describe the dynamics of two-dimensional (2D) modes which decouples from 3D waves.

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Inertial waves are a ubiquitous feature of incompressible fluids under rapid rotation [1]. Although much is known about their initial excitation, still little is understood about their nonlinear interactions. The study of (strong) rotating flows is of interest for a wide range of problems, ranging from engineering (turbomachinery) to geophysics (oceans, earth's atmosphere, gaseous planets) and weather prediction. Rotation is often coupled with other dynamical factors, therefore it is important to isolate rotation to understand precisely its effects. The strength of the Coriolis force, measured in terms of the advection term in the Navier-Stokes (NS) equations, is given by the dimensionless Rossby number $R_o=U/(L\Omega)$, where U is a typical velocity, L a typical length scale, and Ω the rotation rate. Typical values for large-scale planetary flows [2] are 0.05–0.2.

Several experiments have been performed on turbulent fluids under rapid rotation [3,4]. One of the main results observed is that the rapid rotation leads to two-dimensional behavior of an initial homogeneous isotropic turbulence. Evidence of the two-dimensional behavior is revealed through anisotropic spectra where energy is preferentially accumulated in the direction perpendicular to the rotation axis. Recently, energy spectrum $E(k)\sim k^{-2}$ has been experimentally observed [4], instead of the Kolmogorov spectrum $\sim k^{-5/3}$ for nonrotating fluids. This experimental spectrum is interpreted as the result of an inverse cascade of two-dimensional (2D) turbulence.

Turbulent fluids under strong rotation have been widely investigated through numerical simulations [5,6], closure models [7], heuristic descriptions [8], and studies of weakly nonlinear resonant waves [9,10]. The tendency toward a two-dimensional behavior of the turbulence has been observed but surprisingly there is no theoretical prediction and no measure of the scaling law of *anisotropic* spectra. The nonlinear mechanism leading to such a state is still not well understood; neither are the different scalings for the energy spectrum obtained numerically when a forcing is applied at intermediate scale [6]. Important questions concern the origin of the mechanism leading to an interaction between the 2D and the 3D states, and the direction of the energy cascade [6].

Strong rotation introduces in the problem a small parameter, proportional to R_o, from which it is possible to expand the NS equations in the framework of weak turbulence. In this paper, we present such an approach for inertial waves in incompressible rotating fluids. Weak turbulence provides a useful paradigm to understand several challenging problems of turbulence [11,12]; this formalism leads to wave kinetic equations (WKE) that describe the evolution of kinetic energy and helicity spectra. The analysis of the WKE derived in this paper confirms the tendency toward anisotropy in such flows, and leads for the first time, to the best of our knowledge, to exact predictions in terms of anisotropic power law spectra and energy cascade direction.

The Navier-Stokes equations for incompressible flows in a rotating frame read

$$\partial_t \mathbf{w}-2(\mathbf{\Omega}\cdot\nabla)\mathbf{u}=(\mathbf{w}\cdot\nabla)\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{w}+\nu\nabla^2\mathbf{w}, \qquad (1)$$

where $\mathbf{u}$ is the velocity field ($\nabla\cdot\mathbf{u}=0$), $\mathbf{w}$ the vorticity ($\mathbf{w}=\nabla\times\mathbf{u}$), $\mathbf{\Omega}=\Omega\hat{\mathbf{e}}_\parallel$ ($|\hat{\mathbf{e}}_\parallel|=1$), and ν the kinematic viscosity. We assume that the rotation is fast ($R_o\ll 1$), which implies that velocity $\mathbf{v}$ and vorticity $\mathbf{w}$ are much smaller in magnitude than $\mathbf{\Omega}$. We will therefore substitute in the previous equation, $\mathbf{u}\rightarrow\epsilon\mathbf{u}$ and $\mathbf{w}\rightarrow\epsilon\mathbf{w}$, where ϵ is a small parameter ($0<\epsilon\ll 1$). The dispersion law in Fourier space, setting $\epsilon=0$, leads to $\partial_t\mathbf{w}_\mathbf{k}=-2\Omega k_\parallel(\hat{\mathbf{e}}_k\times\mathbf{w}_\mathbf{k})/k=is\omega_k\mathbf{w}_\mathbf{k}$, with $\omega(\mathbf{k})=\omega_k=2\Omega k_\parallel/k$, $s=\pm 1$, $\mathbf{w}_\mathbf{k}=is\hat{\mathbf{e}}_k\times\mathbf{w}_\mathbf{k}$, and where wave vector $\mathbf{k}=k\hat{\mathbf{e}}_k=\mathbf{k}_\perp+k_\parallel\hat{\mathbf{e}}_\parallel$ ($k=|\mathbf{k}|$, $k_\perp=|\mathbf{k}_\perp|$, $|\hat{\mathbf{e}}_k|=1$). This corresponds to dispersive transverse circularly polarized (helical) waves with s being the wave polarity.

We will adopt the Eulerian formalism [13] and choose a complex helicity decomposition for inertial waves whose convenience is now well recognized [7,9,10,14–19]. A supplementary advantage of such a decomposition is that it renders projection operators, inherent to a description of incompressible flows, less cumbersome. The end result of such an approach is a set of integrodifferential equations for the spectral density of the invariants of Eq. (1) in the inviscid case, namely, the energy $E(\mathbf{k})$ and helicity $H(\mathbf{k})$ spectra. The helicity decomposition

$$\mathbf{h}^s(\mathbf{k})\equiv\mathbf{h}^s_\mathbf{k}=(\hat{\mathbf{e}}_k\times\hat{\mathbf{e}}_\parallel)\times\hat{\mathbf{e}}_k+is(\hat{\mathbf{e}}_k\times\hat{\mathbf{e}}_\parallel), \qquad (2)$$

has the following properties: $\mathbf{h}^s_\mathbf{k}\cdot\mathbf{h}^{s'}_\mathbf{k}=(2k_\perp^2/k^2)\delta_{-s's}$, $is(\hat{\mathbf{e}}_k\times\mathbf{h}^s_\mathbf{k})=\mathbf{h}^s_\mathbf{k}$, $\mathbf{k}\cdot\mathbf{h}^s_\mathbf{k}=0$, $\mathbf{h}^{-s}_\mathbf{k}=\mathbf{h}^s_{-\mathbf{k}}$. We project the velocity on the basis of helical modes, $\mathbf{u_k}=\Sigma_s a^s(\mathbf{k})e^{is\omega_k t}\mathbf{h^s_k} \equiv \Sigma_s a^s_{\mathbf{k}}e^{is\omega_k t}\mathbf{h^s_k}$, where $a^s_{\mathbf{k}}$ is the modal amplitude in the interaction representation for which, in the linear approximation ($\epsilon=0$), $\partial_t a^s_{\mathbf{k}}=0$; thus, the weak nonlinearities will modify only slowly in time the inertial-waves amplitude. We also have $\mathbf{w_k}=k\Sigma_s s a^s_{\mathbf{k}}e^{is\omega_k t}\mathbf{h^s_k}$. We introduce expressions of the fields into the NS equations written in Fourier space, and we multiply it by vector $\mathbf{h}^s_{-\mathbf{k}}$ to obtain

$$\partial_t a^s_{\mathbf{k}}=\epsilon \sum_{s_p s_q}\int L^{s s_p s_q}_{-\mathbf{kpq}} a^{s_p}_{\mathbf{p}} a^{s_q}_{\mathbf{q}} e^{-i g_{k,pq} t}\delta_{k,pq} d_{pq}, \tag{3}$$

with interaction operator $L^{s s_p s_q}_{\mathbf{kpq}} \equiv L^I$ given by:

$$L^I=\left(\frac{i s k s_p p}{2k_\perp^2}\right)[(\mathbf{q}\cdot \mathbf{h}^{s_p}_{\mathbf{p}})(\mathbf{h}^{s_q}_{\mathbf{q}}\cdot \mathbf{h}^{s}_{\mathbf{k}})-(\mathbf{p}\cdot \ \mathbf{h}^{s_q}_{\mathbf{q}})(\mathbf{h}^{s_p}_{\mathbf{p}}\cdot \mathbf{h}^{s}_{\mathbf{k}})],$$

and $g_{k,pq}=s\omega_k-s_p\omega_p-s_q\omega_q$ (whereas $g_{kpq}=s\omega_k+s_p\omega_p+s_q\omega_q$), $\delta_{k,pq}=\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})$, $d_{pq}=d\mathbf{p}d\mathbf{q}$. The fundamental Eq. (3) contains an exponentially oscillating term essential to the asymptotic closure: weak turbulence deals with variations of spectral densities at very large time, i.e.,for nonlinear transfer times much greater than the wave period. Consequently, most of the nonlinearities will be destroyed by phase mixing and only the resonance terms will survive. The resonance condition for inertial waves corresponds to relations $\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{0}$ and $sk_\parallel/k+s_p p_\parallel/p+s_q q_\parallel/q=0$, which can also be written as

$$\frac{s_p p - sk}{s_q\omega_q}=\frac{s_q q - s_p p}{s\omega_k}=\frac{sk-s_q q}{s_p\omega_p}. \tag{4}$$

These relations will help simplify the WKE and give a proof of the conservation of their ideal invariants as well.

We now take an orthonormal vector basis local to each triad [9,18,19] as follows: $\hat{\mathbf{O}}^1(\mathbf{p})=\hat{\mathbf{n}}\times\hat{\mathbf{e}}_p, \hat{\mathbf{O}}^2(\mathbf{p})=\hat{\mathbf{n}}, \hat{\mathbf{O}}^3(\mathbf{p})=-\hat{\mathbf{e}}_{\mathbf{p}}$ where $\hat{\mathbf{e}}_{\mathbf{p}}=\mathbf{p}/|\mathbf{p}|$ and $\hat{\mathbf{n}}=(\mathbf{k}\times\mathbf{p})/|\mathbf{k}\times\mathbf{p}|=(\mathbf{p}\times\mathbf{q})/|\mathbf{p}\times\mathbf{q}|=(\mathbf{q}\times\mathbf{k})/|\mathbf{q}\times\mathbf{k}|$. Vector $\hat{\mathbf{n}}$ is normal to any vector of triad ($\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$) and it changes sign by interchanging $\mathbf{p}$ and $\mathbf{q}$ but not by cyclic permutation. One introduces vectors $\Xi^{s_p}(\mathbf{p})\equiv\Xi^{s_p}_{\mathbf{p}}=\hat{\mathbf{O}}^1(\mathbf{p})+is_p\hat{\mathbf{O}}^2(\mathbf{p})$ and we define rotation angle Φ_p such that $\cos\Phi_p=\hat{\mathbf{n}}\cdot(\hat{\mathbf{e}}_{\mathbf{p}}\times\hat{\mathbf{e}}_\parallel)$ and $\sin\Phi_p=\hat{\mathbf{n}}\cdot[(\hat{\mathbf{e}}_{\mathbf{p}}\times\hat{\mathbf{e}}_\parallel)\times\hat{\mathbf{e}}_{\mathbf{p}}]$. It leads to relation $\mathbf{h}^s_{\mathbf{k}}=(k_\perp/k)\Xi^s_{\mathbf{k}}e^{-is\Phi_k}$. With $c^s(\mathbf{k})\equiv c^s_{\mathbf{k}}=(k_\perp/k)a^s_{\mathbf{k}}$, we have

$$\partial_t c^s_{\mathbf{k}}=\epsilon\sum_{s_p s_q}\int (s_q q - s_p p) M^{s s_p s_q}_{-\mathbf{kpq}} c^{s_p}_{\mathbf{p}} c^{s_q}_{\mathbf{q}} e^{-i g_{k,pq}t}\delta_{k,pq} d_{pq} \tag{5}$$

with

$$M^{s s_p s_q}_{\mathbf{kpq}}=\frac{i}{4}(sk+s_p p+s_q q)\frac{\sin\alpha_k}{k}e^{-i(s\Phi_k+s_p\Phi_p+s_q\Phi_q)}, \tag{6}$$

where α_k refers to the angle between $\mathbf{p}$ and $\mathbf{q}$ in triangle $\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{0}$. The local decomposition allows one to concentrate concisely complex information in a unique exponential function which will help simplify notably the derivation of the WKE.

We define energy density tensor $e^{s'}(\mathbf{k}')$ for homogeneous turbulence, such that $\langle c^s(\mathbf{k})c^{s'}(\mathbf{k}')\rangle\equiv e^{s'}(\mathbf{k}')\delta(\mathbf{k}+\mathbf{k}')\delta_{ss'}$, for which we shall write a "closure" equation. The presence of delta function $\delta_{ss'}$ means that the correlations between opposite polarities have no long-time influence in the weak turbulence regime. The derivation of the WKE is a technical and lengthy but classical calculation of weak turbulence (see, e.g., [12,13]), which will be presented elsewhere. In this paper attention is rather focused on the main properties of the WKE, which to our knowledge are given for the first time here:

$$\begin{aligned}\partial_t e^s(\mathbf{k})=\frac{4\pi\epsilon^2}{s\omega_k}\sum_{s_p s_q}\int (s_q q & \\ -s_p p)^2 |M^{s s_p s_q}_{\mathbf{kpq}}|^2 \delta(g_{kpq})\delta_{kpq}[s\omega_k e^{s_p}(\mathbf{p})e^{s_q}(\mathbf{q}) & \\ +s_p\omega_p e^s(\mathbf{k})e^{s_q}(\mathbf{q})+s_q\omega_q e^s(\mathbf{k})e^{s_p}(\mathbf{p})]d_{pq}. &\end{aligned} \tag{7}$$

Equation (7) it is the first main result of this paper; it describes statistical properties of inertial wave turbulence. Matrix $M^{s s_p s_q}_{\mathbf{kpq}}$ is taken as a modulus, which means that the complex information concentrated in the exponential function does not enter into account in the dynamics. Note that the resonance condition appears as delta function $\delta(g_{kpq})$. As expected, the WKE conserve in detail (for each triadic interaction) ideal invariants, i.e., energy $E(t)\equiv\int\Sigma_s e^s(\mathbf{k})d\mathbf{k}$ and helicity $H(t)\equiv\int\Sigma_s s k e^s(\mathbf{k})d\mathbf{k}$.

After simple manipulations, in particular to introduce spectra $E(\mathbf{k})$ and $H(\mathbf{k})$, we obtain the general expression of the WKE at the level of three-wave interactions:

$$\partial_t\left\{\begin{matrix}E(\mathbf{k})\\ H(\mathbf{k})\end{matrix}\right\}=\frac{\pi\epsilon^2}{8}\sum_{s s_p s_q}\int\left(\frac{s_q q - s_p p}{\omega_k}\right)^2 (sk+s_p p+s_q q)^2\left(\frac{\sin\alpha_k}{k}\right)^2\delta(g_{kpq})\delta_{kpq} s\omega_k s_p\omega_p\left\{\begin{matrix}X_E\\ X_H\end{matrix}\right\}d_{pq} \tag{8}$$

with

$$\left\{\begin{matrix}X_E\\ X_H\end{matrix}\right\}=\left\{\begin{matrix}E(\mathbf{q})[E(\mathbf{k})-E(\mathbf{p})]+(H(\mathbf{q})/s_q q)[H(\mathbf{k})/sk-H(\mathbf{p})/s_p p]\\ sk\{E(\mathbf{q})[H(\mathbf{k})/sk-H(\mathbf{p})/s_p p]+[H(\mathbf{q})/s_q q)(E(\mathbf{k})-E(\mathbf{p})]\}\end{matrix}\right\}. \tag{9}$$

Several properties are observed. First, we see that an initial state with zero helicity will not generate any helicity at any scale. Second, we observe that there is no coupling between helicity states associated with wave vectors $\mathbf{p}$ and $\mathbf{q}$, when they are collinear ($\sin\alpha_k = 0$). Third, there is no coupling between helicity states associated with these vectors whenever magnitudes p and q are equal if their associated polarities s_p and s_q are also equal. This property holds for both energy and kinetic helicity; it is a standard property of helical waves [9,15,18,19]. Fourth, a strong helical perturbation, localized initially in a narrow band of wave numbers, will lead therefore to a weak transfer of energy and helicity [15,20,21].

Because anisotropic turbulence prevails in several numerical simulations [5,6], it is of interest to investigate the local interaction limit (equilateral triadic wave coupling $k \approx p \approx q$) of Eqs. (8,9) in order to understand precisely the primary dynamics leading to anisotropic turbulence. Then the resonance condition (4) reads $(s_p - s)/s_q q_\parallel \approx (s_q - s_p)/s k_\parallel \approx (s - s_q)/s_p p_\parallel$. From equations (8,9), we see that only interactions between two waves ($\mathbf{p}$ and $\mathbf{q}$) with opposite polarities ($s = s_p = -s_q$ or $s = -s_p = s_q$) will contribute significantly to the nonlinear dynamics. It implies that either $q_\parallel \approx 0$ or $p_\parallel \approx 0$, which means that only a small transfer is allowed along $\mathbf{\Omega}$. In other words, local nonlinear interactions lead to anisotropic turbulence where small scales are preferentially generated perpendicular to the external rotation vector. Note the similarity with electrically conducting magneto hydro dynamic fluids for which the presence of a strong uniform magnetic field prevents any energy transfer along it [22].

The previous reasoning allows us to consider the anisotropic limit, i.e., the $k_\perp \gg k_\parallel$ limit. This means that, for example, we assume that the turbulence is rather generated initially by a source in a limited band of (large) scales. Local interactions will therefore dominate and they will lead essentially to anisotropic turbulence, i.e., structures elongated along the rotation axis like the vortices observed experimentally in [4]. At leading order in $k_\parallel/k_\perp$, and for an inertial wave turbulence that is axially symmetric with respect to the rotation vector, the simplified WKE read

$$\partial_t \begin{Bmatrix} E_k \\ H_k \end{Bmatrix} = \frac{\Omega^2 \epsilon^2}{4} \sum_{s s_p s_q} \int \frac{s k_\parallel s_p p_\parallel}{k_\perp^2 p_\perp^2 q_\perp^2} \left(\frac{s_q q_\perp - s_p p_\perp}{\omega_k} \right)^2 (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \sin\theta\, \delta(g_{kpq})\, \delta_{k_\parallel p_\parallel q_\parallel} \begin{Bmatrix} E_q(p_\perp E_k - k_\perp E_p) + (p_\perp s H_k/k_\perp - k_\perp s_p H_p/p_\perp) s_q H_q/q_\perp \\ s k_\perp [E_q(p_\perp s H_k/k_\perp - k_\perp s_p H_p/p_\perp) + (p_\perp E_k - k_\perp E_p) s_q H_q/q_\perp] \end{Bmatrix} dp_\perp dq_\perp dp_\parallel dq_\parallel, \tag{10}$$

with θ the angle between $\mathbf{k}_\perp$ and $\mathbf{p}_\perp$ in triangle $\mathbf{k}_\perp + \mathbf{p}_\perp + \mathbf{q}_\perp = \mathbf{0}$, $\omega_k = 2\Omega k_\parallel/k_\perp$ and $E_k = E(k_\perp, k_\parallel) = 2\pi k_\perp E(\mathbf{k}_\perp, k_\parallel)$, $H_k = H(k_\perp, k_\parallel) = 2\pi k_\perp H(\mathbf{k}_\perp, k_\parallel)$.

Exact solutions of Eqs. (10) as power laws can be found by applying the Kuznetsov-Zakharov conformal transformation [11] which is a 2D generalization of the Zakharov transformation. The most interesting solutions are those for which the flux is finite (instead of being null, as in the thermodynamic solutions). These exact solutions, the Kuznetsov-Zakharov-Kolmogorov (KZK) spectra, read

$$E(k_\perp, k_\parallel) \sim k_\perp^{-5/2} k_\parallel^{-1/2}, \quad H(k_\perp, k_\parallel) \sim k_\perp^{-3/2} k_\parallel^{-1/2}. \tag{11}$$

The anisotropic limit is the only case where the theory works well in the sense that the collision integral does not suffer from infrared divergences which require corrections to the spectra, e.g., logarithmic [23]. Although it is possible to derive the exact expressions of the Kolmogorov constants appearing in front of spectra (11), it is difficult to give them precise values since they depend on the cutoffs introduced by the anisotropic limit [24]. However, the sign of the energy transfer can be computed for a reasonable range of cutoffs; it is found positive, hence a direct energy cascade. These latter two exact results, which cannot be found by simple heuristic descriptions, are particularly significant since they suggest experimental measurements to compare with.

On the other hand, the KZK spectra can be obtained phenomenologically. Dimensional analysis leads to relation $\bar{\epsilon} \sim u^2/\tau_{tr} \sim E(k_\perp, k_\parallel) k_\perp k_\parallel/\tau_{tr}$, where $\bar{\epsilon}$ is the mean rate of energy dissipation per unit of mass and τ_{tr} is the transfer time whose form is given by the WKE. We have $\tau_{tr} \sim \tau_{NL}^2/\tau_\Omega$, with τ_{NL} the nonlinear characteristic time and τ_Ω the inertial-wave period. Anisotropic turbulence leads to scaling $\tau_{NL} \sim \ell_\perp/u \sim (k_\perp u)^{-1}$, and $\tau_\Omega \sim k_\perp/\Omega k_\parallel$. Finally, we obtain spectrum $E(k_\perp, k_\parallel) \sim (\bar{\epsilon}\Omega)^{1/2} k_\perp^{-5/2} k_\parallel^{-1/2}$. Note that if we forget the anisotropic hypothesis and assume $k_\perp \approx k_\parallel \approx k$, we recover the earlier prediction $E(k) \sim (\bar{\epsilon}\Omega)^{1/2} k^{-2}$ for the 1D isotropic energy spectrum [8], which is moreover a solution of isotropic direct interaction approximation equations [25].

The primary dynamics leading to anisotropic turbulence may stop at some point since nonlocal interactions develop as well. According to direct numerical simulations (DNS), anisotropy is generated and preserved. It seems therefore that the possible balance between local and nonlocal interactions does not lead to an isotropization of the turbulence. Analysis of resonance condition (4) shows indeed that strongly nonlocal interactions lead also to anisotropic turbulence [10]. Therefore, our theoretical analysis confirms that there is a global nonlinear tendency to develop and maintain anisotropy.

Inertial-wave turbulence theory is an asymptotic theory based on a time scale separation $\tau_{tr} \gg \tau_\Omega$. (For strong turbulence, we have $\tau_{tr} \sim \tau_{NL} \sim \tau_\Omega$.) The consequence is that the theory is *not* valid in the entire **k** space. Evaluation of the transfer time gives (see above) $\tau_{tr} \sim \epsilon^{-2}\Omega k_\parallel^{1/2} k_\perp^{-3/2}$ [26]. The time scale separation condition (with $\Omega \sim 1$) leads to $k_\parallel \gg \epsilon^{4/3} k_\perp^{5/3}$. The previous relation combined with the anisotropic assumption defines the domain of validity in **k** space of our theory at the level of three-wave interactions. We have a nonuniform validity of the WKE which means, in particular, that the theory is not valid for too small values of $k_\parallel$ or too large values of $k_\perp$. It is important to note that inside the prohibited region, other interactions like higher-order processes (four-wave interactions, . . .) have to be taken into account (see, e.g., [12,27]). In particular, Eq. (10) shows that the nonlinear transfer, for energy and helicity, decreases (linearly) with $k_\parallel$. For forbidden value $k_\parallel = 0$ the transfer is exactly null. As previously mentioned [6,10], the 2D (geostrophic or slow) modes decouple from the 3D inertial waves. Such decoupling underlies the validity of quasigeostrophic models, e.g., for the atmosphere or the oceans [28]. Forced DNS [6] show the generation of slow modes. From the present work, we see that weak turbulence at the lowest order cannot describe such an observation; however higher-order processes could play a significant role [6].

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